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视网膜氧分布与脑红蛋白作用模型解的存在唯一性

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摘要: 研究视网膜中氧分布与脑红蛋白作用的数学模型, 该模型包含了4组相互耦合的反应扩散方程组. 先通过运用 Banach 不动点定理, 抛物型方程的 L^p 估计证明了模型的局部解的存在唯一性, 然后利用延拓方法得到了整体解的存在唯一性.

关键词: 视网膜; 氧分布; 局部解; 整体解; 存在唯一性

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Existence and Uniqueness of Global Solution for a Model of Retinal Oxygen Distribution and the Role of Neuroglobin

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Abstract: A study is conducted on a mathematical model of retina oxygen distribution and the role of neuroglobin, which contains four sets of mutually coupled reaction diffusion equations. The existence and uniqueness of the model in the local solution is proved by using the Banach Fixed Point Theorem, applying L^p -theory of parabolic equation. And then the existence and uniqueness of the global solution is obtained by using the extension method.

Key words: retina; oxygen distribution; local solution; global solution; existence and uniqueness

视网膜居于眼球壁的内层, 是一层透明的薄膜. 它由色素上皮层和视网膜感觉层组成, 色素上皮层与脉络膜紧密相连, 具有支持和营养光感受器细胞、遮光、散热以及再生和修复等作用, 是检测视觉信息的视觉组织. 视网膜的能量供给需要眼部组织有最高的氧消耗率, 作为最大的无血管收集组织, 玻璃体、晶状体和角膜满足了视网膜的能量供给需求. 而这些组织的完整性取决于分子氧对细胞有机基质氧化时产生能量的不断消耗. 若要了解视网膜, 全面了解氧的浓度变化对视网膜的影响是必要的^[1-2]. 当人类视网膜上的氧分布还没有被测量时, 人们测量了各种哺乳动物的视网膜氧分布, 其中包括白鼠、兔子和猴子等. 学者发现在黑暗的环境下感光器消耗氧气的速度是在光照条件下消耗氧气的两倍^[3-7]. 在黑暗环境下氧分布减少, 即视网膜在黑暗环境下更容易缺氧. 无论是限制氧化代谢还是增加活性氧产生

方面, 缺氧均对细胞有害, 表现为破坏氧化还原电位, 导致氧化应激和损伤等^[8]. 例如, 大多数视网膜失明是一种与血管成分有关的疾病, 而中断视网膜内氧供应是该疾病产生的关键因素; 又如, 视网膜循环障碍引起组织缺氧, 造成血管增生, 从而导致糖尿病视网膜病和早产儿视网膜病^[9]. 故研究视网膜的氧分布对人体健康很有必要. 而在视网膜组织中, 脑红蛋白有运输和存储氧气的作用, 可以增加氧摄取和预防缺氧. 因此, 深入研究视网膜中氧分布及脑红蛋白在此过程的具体作用有非常重要的现实意义.

D. Y. Yu和 S. J. Cringle等学者^[10-11]于2002年提出过一系列关于视网膜氧分布的数学模型, 模型假设氧浓度处于稳态, 并且呈现在跨越视网膜的宽度的沿径向的一维结构域上, 即每个模型层的吸氧速率是恒定的. 2016年, P. A. Roberts, E. A. Gaffney及P. J. Luthert等学者^[12]认为视网膜结构遵循一定的数学规

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律,在一维中可表述为偏微分方程模型,从而提出了一个全新的视网膜氧分布与脑红蛋白作用的模型,在该模型中,P. A. Roberts等学者进行了数值模拟,对数值解进行渐近分析.具体模型如下:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - \frac{Qc}{\gamma + c} + k_2v - k_1nc, \quad 0 \leq x \leq L, t > 0, \quad (1)$$

$$c(0, t) = c_c, \quad \frac{\partial c}{\partial x}(L, t) = 0, \quad (2)$$

$$c(x, 0) = c_0(x), \quad (3)$$

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} + k_4u + \alpha k_2v - k_3n - \alpha k_1nc, \quad 0 \leq x \leq L, t > 0, \quad (4)$$

$$\frac{\partial n}{\partial x}(0, t) = \frac{\partial n}{\partial x}(L, t) = 0, \quad (5)$$

$$n(x, 0) = n_0(x), \quad (6)$$

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + k_3n - k_4u, \quad 0 \leq x \leq L, t > 0, \quad (7)$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0, \quad (8)$$

$$u(x, 0) = u_0(x), \quad (9)$$

$$\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + \alpha k_1nc - \alpha k_2v, \quad 0 \leq x \leq L, t > 0, \quad (10)$$

$$\frac{\partial v}{\partial x}(0, t) = \frac{\partial v}{\partial x}(L, t) = 0, \quad (11)$$

$$v(x, 0) = v_0(x). \quad (12)$$

其中, $c(x, t)$ 、 $n(x, t)$ 、 $u(x, t)$ 、 $v(x, t)$ 分别表示氧气、 Ngb (脑红蛋白)、 $Ngb-His$ (脑红蛋白与组氨酸)以及 $Ngb-O_2$ (脑红蛋白与氧气)的质量浓度. Q 表示视网膜组织最大摄氧量, γ 表示缺氧阈值(区域中氧浓度低于此值被认为是缺氧), c_c 表示脉络膜毛细血管层的氧浓度, $\alpha, k_1, k_2, k_3, k_4$ 为常数.

参考文献[13-19]的方法,拟对文献[12]中的模型进行严格的数学分析,再根据生物学和医学原理,本文做出以下假设:

(a) $0 \leq \alpha \leq 1$ (其中 α 表示的是脉络膜毛细血管层的氧浓度与视网膜处 Ngb 的平均浓度的比值);

(b) $c_0(x), n_0(x), u_0(x), v_0(x) \in D_p(0, L)$, 且 $c_0(x), n_0(x), u_0(x), v_0(x) \geq 0$.

本文的主要结论如下:

定理1 满足假设(a)和(b)的条件下,对 $\forall t \geq 0$,问题(1)~(12)存在唯一的整体解.

1 预备引理

下面将介绍一些引理,首先引入一些记号.

(1) 记 $Q_T = \{(r, t) : 0 < r < L, 0 < t < T\}$, $\bar{Q}_T$ 是 Q_T 的闭包.

(2) $W_p^{2,1}(Q_T) = \{u, v \in L^p(Q_T) : u_t, v_t, \nabla u, \nabla^2 u, \nabla v,$

$\nabla^2 v \in L^p(Q_T)\}$ 且规定

$$\|u\|_{W_p^{2,1}(Q_T)} = \sum_{|m|+2k \leq 2} \|\partial_x^m \partial_t^k u\|_{L^p}.$$

(3) $D_p(0, L) = \{u(x, 0) = \phi(x) | u(x, t) \in W_p^{2,1}\}$, 且规定

$$\|\phi\|_{D_p(0, L)} = \inf \left\{ T^{-\frac{1}{p}} \|u\|_{W_p^{2,1}(Q_T)}, u(., x) = \phi(x) \right\}.$$

(4) 对 $p > \frac{5}{2}$, 记 $D_p(0, 1)$ 为 $t = 0$ 时 $W_p^{2,1}(Q_T)$ 的迹空间, i.e. $\phi \in D_p(0, 1)$ 当且仅当 $\exists u \in W_p^{2,1}(Q_T)$, 使得 $u(., 0) = \phi$. 定义 $D_p(0, 1)$ 中的范数如下:

$$\|\phi\|_{D_p(0, L)} = \left\{ T^{-\frac{1}{p}} \|u\|_{W_p^{2,1}(Q_T)} | u \in W_p^{2,1}(\Omega_T), u(., x) = \phi \right\}.$$

由于 $p > \frac{5}{2}$, $W_p^{2,1}(Q_T)$ 连续嵌入到 $C(\Omega_T)$ (见文献[20]), 上面的定义是有意义的. 而且, 显然如果 $\phi \in W^{2,p}(0, 1)$, 则 $\phi \in D_p(0, 1)$ 且 $\|\phi\|_{D_p(0, L)} \leq \|\phi\|_{W^{2,p}(0, 1)}$.

引理1^[21] 假设 D 是一个正常数, $a(z, \tau), b(z, \tau)$ 是定义在区间 $\bar{Q}_T$ 上的有界连续函数, $f(z, \tau) \in L^p(Q_T)$, $\phi(z, \tau) \in C^1[0, T]$, 且对 $1 < p < \infty, c_0 \in D_p(0, L)$. 令 $Bu = \alpha \frac{\partial u}{\partial n} + \beta(x, t)u$, 其中(1) $\alpha = 0, \beta = 1$, (2) $\alpha = 1, \beta \geq 0$, 则初边值问题

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + a(z, \tau) \frac{\partial c}{\partial z} + b(z, \tau)c + f(z, \tau), \quad (13)$$

$$0 < z \leq 1, 0 < t \leq T, \quad z = 0, 1 : Bc = \phi, 0 \leq \tau \leq T, \quad (14)$$

$$c(z, 0) = c_0(z), 0 \leq z \leq 1, \quad (15)$$

有唯一解 $c(z, \tau) \in W_p^{2,1}(Q_T)$, 且

$$\|c\|_{W_p^{2,1}(Q_T)} \leq C_p(T) (\|c_0\|_{D_p(0, L)} + \|\phi\|_{W^{1,p}(0, T)} + \|f\|_p).$$

其中, $C_p(T)$ 是一个依赖于 $p, T, \|a\|_\infty, \|b\|_\infty$ 的常数, 且对任意的有界集 $T, C_p(T)$ 是有界的.

2 局部解的存在唯一性

本文将运用Banach不动点定理证明问题式(1)~式(12)存在局部唯一解. 对于给定的 T 和正整数 M , 引进度量空间 $(X_T, d) : X_T$ 是由所有的向量函数 $(c(x, t), n(x, t), u(x, t), v(x, t)) (0 \leq x \leq L, 0 \leq t \leq T)$ 组成, 它们满足如下条件

$$c, n, u, v \in C(\bar{Q}_T), 0 \leq c, n, u, v \leq M.$$

定义 X_T 中的度量空间 d 为

$$d((c_1, n_1, u_1, v_1), (c_2, n_2, u_2, v_2)) = \max_{\bar{Q}_T} |c_1(x, t) - c_2(x, t)| + \max_{\bar{Q}_T} |n_1(x, t) - n_2(x, t)| + \max_{\bar{Q}_T} |u_1(x, t) - u_2(x, t)| + \max_{\bar{Q}_T} |v_1(x, t) - v_2(x, t)|.$$

易知度量空间 (X_T, d) 是一个完备的度量空间.

对任意的 $(c(x, t), n(x, t), u(x, t), v(x, t)) \in X_T$, 定义一个映射 $F: (c(x, t), n(x, t), u(x, t), v(x, t)) \mapsto (\tilde{c}(x, t), \tilde{n}(x, t), \tilde{u}(x, t), \tilde{v}(x, t))$, 考虑如下问题:

$$\frac{\partial \tilde{c}}{\partial t} = \frac{\partial^2 \tilde{c}}{\partial x^2} - \frac{Q\tilde{c}}{\gamma + c} + k_2 v - k_1 n \tilde{c}, \quad 0 \leq x \leq L, t > 0, \quad (16)$$

$$\tilde{c}(0, t) = c_c, \quad \frac{\partial \tilde{c}}{\partial x}(L, t) = 0, \quad (17)$$

$$\tilde{c}(x, 0) = c_0(x), \quad (18)$$

$$\frac{\partial \tilde{n}}{\partial t} = D \frac{\partial^2 \tilde{n}}{\partial x^2} + k_4 u + \alpha k_2 v - k_3 \tilde{n} - \alpha k_1 \tilde{n} c, \quad 0 \leq x \leq L, t > 0, \quad (19)$$

$$\frac{\partial \tilde{n}}{\partial x}(0, t) = \frac{\partial \tilde{n}}{\partial x}(L, t) = 0, \quad (20)$$

$$\tilde{n}(x, 0) = n_0(x), \quad (21)$$

$$\frac{\partial \tilde{u}}{\partial t} = D \frac{\partial^2 \tilde{u}}{\partial x^2} + k_3 n - k_4 \tilde{u}, \quad 0 \leq x \leq L, t > 0, \quad (22)$$

$$\frac{\partial \tilde{u}}{\partial x}(0, t) = \frac{\partial \tilde{u}}{\partial x}(L, t) = 0, \quad (23)$$

$$\tilde{u}(x, 0) = u_0(x), \quad (24)$$

$$\frac{\partial \tilde{v}}{\partial t} = D \frac{\partial^2 \tilde{v}}{\partial x^2} + \alpha k_1 n c - \alpha k_2 \tilde{v}, \quad 0 \leq x \leq L, t > 0, \quad (25)$$

$$\frac{\partial \tilde{v}}{\partial x}(0, t) = \frac{\partial \tilde{v}}{\partial x}(L, t) = 0, \quad (26)$$

$$\tilde{v}(x, 0) = v_0(x). \quad (27)$$

首先证明 F 映 X_T 到 X_T 自身.

(1) 由上下解原理, 可得 $\tilde{c} \geq 0, \tilde{n} \geq 0, \tilde{u} \geq 0, \tilde{v} \geq 0$.

(2) 考虑问题(16)~(18), $k_2 v \in L_p(Q_T)$, 由引理1知它有唯一解 $\tilde{c} \in W_p^{2,1}(Q_T)$, 且满足

$$\|\tilde{c}\|_{W_p^{2,1}(Q_T)} \leq C(T) (\|c_0\|_{D_P(0,L)} + \|c_c\|_{W^{1,p}(0,T)} + \|k_2 v\|_p) \leq C(T)N.$$

N 为依赖于 $\|c_0(x)\|_{D_P(0,L)}$ 和 M 的常数.

(3) 方程(19)的系数项和非齐次项均满足引理1的条件, 由引理1知它有唯一解 $\tilde{n} \in W_p^{2,1}(Q_T)$, 且满足

$$\|\tilde{n}\|_{W_p^{2,1}(Q_T)} \leq C(T) (\|n_0\|_{D_P(0,L)} + \|k_4 u + \alpha k_2 v\|_p) \leq C(T)N.$$

N 为依赖于 $\|n_0(x)\|_{D_P(0,L)}$ 和 M 的常数.

(4) 同理, 由引理1可知, 问题(22)~(24)及(25)~(27)有唯一解 $\tilde{u} \in W_p^{2,1}(Q_T), \tilde{v} \in W_p^{2,1}(Q_T)$. 且满足

$$\|\tilde{u}\|_{W_p^{2,1}(Q_T)} \leq C(T)N, \quad \|\tilde{v}\|_{W_p^{2,1}(Q_T)} \leq C(T)N.$$

综上所述, 若取 $N > 0$, 则当 $T > 0$ 充分小时, $C_p(T)$ 是有界的, $C(T)N \leq M$, 有 $\|\tilde{c}\|_{W_p^{2,1}(Q_T)} \leq M$, $\|\tilde{n}\|_{W_p^{2,1}(Q_T)} \leq M, \|\tilde{u}\|_{W_p^{2,1}(Q_T)} \leq M, \|\tilde{v}\|_{W_p^{2,1}(Q_T)} \leq M$. 由 $W_p^{2,1}(Q_T) \subset C^{\alpha, \frac{\alpha}{2}}(Q_T) \left(p > \frac{5}{2}\right)$, 得 $\|\tilde{c}\|_\infty \leq M, \|\tilde{n}\|_\infty \leq M$,

$\|\tilde{u}\|_\infty \leq M, \|\tilde{v}\|_\infty \leq M$. 对任意 $(c(x, t), n(x, t), u(x, t), v(x, t)) \in X_T$, 存在 $(\tilde{c}(x, t), \tilde{n}(x, t), \tilde{u}(x, t), \tilde{v}(x, t)) \in X_T$, 即 F 映 X_T 到 X_T 自身.

接下来证明 F 是压缩映射.

(1) 定义 $\tilde{c}_* = \tilde{c}_1 - \tilde{c}_2$, 由问题(16)~(18), 有

$$\frac{\partial \tilde{c}_*}{\partial t} = \frac{\partial^2 \tilde{c}_*}{\partial x^2} + b_1(x, t) \tilde{c}_* + f_1(x, t), \quad 0 \leq x \leq L, t > 0, \quad (28)$$

$$\tilde{c}_*(0, t) = \frac{\partial \tilde{c}_*}{\partial x}(L, t) = 0, \quad (29)$$

$$\tilde{c}_*(x, 0) = 0, \quad (30)$$

其中

$$b_1(x, t) = -k_1 n_1 - \frac{Q(\gamma + c_1)}{(\gamma + c_1)(\gamma + c_2)}, \quad (31)$$

$$f_1(x, t) = \frac{Q\tilde{c}_1(c_1 - c_2)}{(\gamma + c_1)(\gamma + c_2)} + k_2(v_1 - v_2) - k_1\tilde{c}_2(n_1 - n_2). \quad (32)$$

易知 $b_1(x, t)$ 有界连续, $c, \tilde{c}_2, v, n \in C(\bar{Q}_T)$, 故 $b_1(x, t), f_1(x, t) \in L^p(Q_T)$, 由解的最大模估计, 有

$$\begin{aligned} \|\tilde{c}_*\|_\infty &\leq T \|f_1\|_\infty = \\ &T \left\| \frac{Q\tilde{c}_1(c_1 - c_2)}{(\gamma + c_1)(\gamma + c_2)} + k_2(v_1 - v_2) - k_1\tilde{c}_2(n_1 - n_2) \right\|_\infty \leq \\ &T \left\| \frac{Q\tilde{c}_1(c_1 - c_2)}{(\gamma + c_1)(\gamma + c_2)} \right\|_\infty + \\ &k_2 T \|v_1 - v_2\|_\infty - k_1 T \|\tilde{c}_2\|_\infty \cdot \|n_1 - n_2\|_\infty \leq \\ &TMd. \end{aligned} \quad (33)$$

(2) 定义 $\tilde{n}_* = \tilde{n}_1 - \tilde{n}_2$, 由问题(19)~(21), 有

$$\begin{aligned} \frac{\partial \tilde{n}_*}{\partial t} &= D \frac{\partial^2 \tilde{n}_*}{\partial x^2} - (k_3 + \alpha k_1 c_2) \tilde{n}_* + k_4(u_1 - u_2) + \\ &\alpha k_2(v_1 - v_2) - \alpha k_1 \tilde{n}_1(c_1 - c_2), \end{aligned}$$

化简为

$$\frac{\partial \tilde{n}_*}{\partial t} = D \frac{\partial^2 \tilde{n}_*}{\partial x^2} + b_2(x, t) \tilde{n}_* + f_2(x, t), \quad 0 \leq x \leq L, t > 0, \quad (34)$$

$$\tilde{n}_*(0, t) = \frac{\partial \tilde{n}_*}{\partial x}(L, t) = 0, \quad (35)$$

$$\tilde{n}_*(x, 0) = 0, \quad (36)$$

其中

$$b_2(x, t) = -(k_3 + \alpha k_1 c_2), \quad (37)$$

$$f_2(x, t) = k_4(u_1 - u_2) + \alpha k_2(v_1 - v_2) - \alpha k_1 \tilde{n}_1(c_1 - c_2). \quad (38)$$

易知 $b_2(x, t), f_2(x, t) \in L^p(Q_T)$, 由解的最大模估计, 有

$$\begin{aligned} \|\tilde{c}_*\|_\infty &\leq T \|f_2\|_\infty = \\ &T \|k_4(u_1 - u_2) + \alpha k_2(v_1 - v_2) - \alpha k_1 \tilde{n}_1(c_1 - c_2)\|_\infty \leq \\ &k_4 T \|u_1 - u_2\|_\infty + \alpha k_2 T \|v_1 - v_2\|_\infty - \\ &\alpha k_1 T \|\tilde{n}_1\|_\infty \cdot \|v_1 - v_2\|_\infty \leq \\ &TMd. \end{aligned} \quad (39)$$

(3) 定义 $\tilde{u}_* = \tilde{u}_1 - \tilde{u}_2$, 由问题(22)~(24), 有

$$\frac{\partial \tilde{u}_*}{\partial t} = D \frac{\partial^2 \tilde{u}_*}{\partial x^2} + k_3(n_1 - n_2) - k_4 \tilde{u}_*, \quad 0 \leq x \leq L, t > 0,$$

化简为

$$\frac{\partial \tilde{u}_*}{\partial t} = D \frac{\partial^2 \tilde{u}_*}{\partial x^2} + b_3(x, t) \tilde{u}_* + f_3(x, t), \quad 0 \leq x \leq L, t > 0, \quad (40)$$

$$\tilde{u}_*(0, t) = 0, \quad \frac{\partial \tilde{u}_*}{\partial x}(L, t) = 0, \quad (41)$$

$$\tilde{u}_*(x, 0) = 0, \quad (42)$$

其中

$$b_3(x, t) = -k_4, \quad (43)$$

$$f_3(x, t) = k_3(n_1 - n_2). \quad (44)$$

由解的最大模估计,有

$$\|\tilde{u}_*\|_\infty \leq T \|f_3\|_\infty = T \|k_3(n_1 - n_2)\|_\infty \leq T M d. \quad (45)$$

(4) 定义 $\tilde{v}_* = \tilde{v}_1 - \tilde{v}_2$, 由问题(25)~(27), 有

$$\frac{\partial \tilde{v}_*}{\partial t} = D \frac{\partial^2 \tilde{v}_*}{\partial x^2} + \alpha k_1(n_1 c_1 - n_2 c_2) - \alpha k_2 \tilde{v}_*, \quad 0 \leq x \leq L, t > 0,$$

化简为

$$\frac{\partial \tilde{v}_*}{\partial t} = D \frac{\partial^2 \tilde{v}_*}{\partial x^2} + b_4 \tilde{v}_* + f_4(x, t), \quad 0 \leq x \leq L, t > 0, \quad (46)$$

$$\tilde{v}_*(0, t) = \frac{\partial \tilde{v}_*}{\partial x}(L, t) = 0, \quad (47)$$

$$\tilde{v}_*(x, 0) = 0, \quad (48)$$

其中

$$b_4(x, t) = -\alpha k_2, \quad (49)$$

$$f_4(x, t) = \alpha k_1(n_1 c_1 - n_2 c_2). \quad (50)$$

同理得

$$\begin{aligned} \|\tilde{v}_*\|_\infty &\leq T \|f_4\|_\infty = T \|\alpha k_1(n_1 c_1 - n_2 c_2)\|_\infty = \\ &\alpha k_1 T \|n_1 c_1 - n_1 c_2 + n_1 c_2 - n_2 c_2\|_\infty \leq \\ &\alpha k_1 T \|n_1\|_\infty \|c_1 - c_2\|_\infty + \alpha k_1 T \|c_2\|_\infty \|n_1 - n_2\|_\infty \leq \\ &T M d. \end{aligned} \quad (51)$$

综上, 由(1)~(4), 可得

$$d((\tilde{c}_1, \tilde{n}_1, \tilde{u}_1, \tilde{v}_1), (\tilde{c}_2, \tilde{n}_2, \tilde{u}_2, \tilde{v}_2)) =$$

$$\|\tilde{c}_1 - \tilde{c}_2\|_\infty + \|\tilde{n}_1 - \tilde{n}_2\|_\infty + \|\tilde{u}_1 - \tilde{u}_2\|_\infty + \|\tilde{v}_1 - \tilde{v}_2\|_\infty \leq C(T) M d.$$

因此, 取 T 充分小, 使得 $C(T) M \leq 1$, 则 F 为压缩映射.

由 Banach 不动点定理, 定义在映射 F 上的一个固定点 $(c(x, t), n(x, t), u(x, t), v(x, t)) \in X_T$, 使得对任意的 $0 \leq t \leq T$, 问题(1)~(12)有唯一古典解 $(c(x, t), n(x, t), u(x, t), v(x, t))$, 注意到 T 依赖于 $\|c_0(x)\|_\infty, \|n_0(x)\|_\infty, \|u_0(x)\|_\infty, \|v_0(x)\|_\infty$ 的上确界.

由上述证明可总结为如下定理.

定理2 存在 $T > 0$, 使得所有 $t \in [0, T]$, 问题(1)~(12)存在唯一解, 其中 T 依赖于 $\|c_0(x)\|_\infty,$

$\|n_0(x)\|_\infty, \|u_0(x)\|_\infty, \|v_0(x)\|_\infty$ 的上确界.

3 整体解的存在唯一性

引理2 问题(1)~(12)的解有如下结论

$$c(x, t) \geq 0, \quad n(x, t) \geq 0, \quad u(x, t) \geq 0, \quad v(x, t) \geq 0.$$

证明 将方程组(1)~(12)转化为如下:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - \frac{Qc}{\gamma + c} - k_1 n c + g_1, \quad 0 \leq x \leq L, t > 0, \quad (52)$$

$$c(0, t) = c_c, \quad \frac{\partial c}{\partial x}(L, t) = 0, \quad (53)$$

$$c(x, 0) = c_0(x), \quad (54)$$

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} - k_3 n - \alpha k_1 n c + g_2, \quad 0 \leq x \leq L, t > 0, \quad (55)$$

$$\frac{\partial n}{\partial x}(0, t) = \frac{\partial n}{\partial x}(L, t) = 0, \quad (56)$$

$$n(x, 0) = n_0(x), \quad (57)$$

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - k_4 u + g_3, \quad 0 \leq x \leq L, t > 0, \quad (58)$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0, \quad (59)$$

$$u(x, 0) = u_0(x), \quad (60)$$

$$\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} - \alpha k_2 v + g_4, \quad 0 \leq x \leq L, t > 0, \quad (61)$$

$$\frac{\partial v}{\partial x}(0, t) = \frac{\partial v}{\partial x}(L, t) = 0, \quad (62)$$

$$v(x, 0) = v_0(x). \quad (63)$$

其中 $g_1 = k_2 v, g_2 = k_4 u + \alpha k_2 v, g_3 = k_3 n, g_4 = \alpha k_1 n c$. 显然, g_1 关于 n, u, v 单调递增, g_2 关于 c, u, v 单调递增, g_3 关于 c, n, v 单调递增, g_4 关于 c, n, u 单调递增, 所以式(52)、(55)、(58)和(61)是一个拟单调递增系统, 且 $(0, 0, 0, 0)$ 是式(52)~(63)的下解, 故

$$c(x, t) \geq 0, \quad n(x, t) \geq 0, \quad u(x, t) \geq 0, \quad v(x, t) \geq 0.$$

引理3 对任意的 $1 < p < \infty$, 存在一个依赖于时间 T 的常数 $C(T)$, 满足

$$\|c\|_\infty \leq C(T), \quad \|n\|_\infty \leq C(T), \quad \|u\|_\infty \leq C(T), \quad \|v\|_\infty \leq C(T).$$

证明 (1) 由式(1)和(10)相加得

$$\begin{aligned} \frac{\partial(c+v)}{\partial t} &\leq D \frac{\partial^2(c+v)}{\partial x^2} + (1-\alpha)k_2 v - (1-\alpha)k_1 n c - \frac{Qc}{\gamma + c} \leq \\ &D \frac{\partial^2(c+v)}{\partial x^2} + k_2(c+v). \end{aligned} \quad (64)$$

将上式(64)两边同时乘以 $(c+v)^k, (k > 0)$ 并在 $Q_T (0 \leq t \leq T)$ 上积分, 可得

$$\begin{aligned} \int_0^t \int_0^L \frac{\partial(c+v)}{\partial t} \cdot (c+v)^k dx dt &\leq \\ D \int_0^t \int_0^L \frac{\partial^2(c+v)}{\partial x^2} \cdot (c+v)^k dx dt &+ k_2 \int_0^t \int_0^L (c+v)^{k+1} dx dt. \end{aligned} \quad (65)$$

对于 $\int_0^t \int_0^L \frac{\partial^2(c+v)}{\partial x^2} \cdot (c+v)^k dx dt$, 通过分部积分可知

$$\int_0^t \int_0^L \frac{\partial^2 c}{\partial x^2} \cdot c^k dx dt = -k \int_0^t \int_0^L |\nabla(c+v)|^2 \cdot (c+v)^{k-1} dx dt \leq 0, \quad (66)$$

从而可得

$$\int_0^t \int_0^L \frac{\partial(c+v)}{\partial t} \cdot (c+v)^k dx dt \leq k_2 \int_0^t \int_0^L (c+v)^{k+1} dx dt. \quad (67)$$

由于

$$\begin{aligned} \int_0^t \int_0^L \frac{\partial(c+v)}{\partial t} \cdot (c+v)^k dx dt &= \frac{1}{k+1} \int_0^L \frac{d}{dt} \int_0^t (c+v)^{k+1} dt dx = \\ &= \frac{1}{k+1} \frac{d}{dt} \int_0^t \int_0^L (c+v)^{k+1} dx dt, \end{aligned} \quad (68)$$

故令 $\eta(x, t) = \int_0^t \int_0^L (c+v)^{k+1} dx dt$ 代入到式(67), 结合式(68), 得

$$\frac{d\eta(x, t)}{dt} \leq C_k(T) \eta(x, t). \quad (69)$$

由Gronwall不等式可得

$$\eta(x, t) \leq C_k(T),$$

因此可得

$$\|c+v\|_{L^p(Q_T)} \leq C_P(T).$$

根据引理2, 有

$$\|c\|_{L^p(Q_T)} \leq C_P(T), \|v\|_{L^p(Q_T)} \leq C_P(T). \quad (70)$$

由引理1, 可得

$$\|c\|_{W_p^{2,1}(\bar{Q}_T)} \leq C_P(T).$$

(2) 由(4), (7)和(10)式相加得

$$\frac{\partial(n+v+u)}{\partial t} = D \frac{\partial^2(n+v+u)}{\partial x^2}. \quad (71)$$

应用引理1有

$$\|n+v+u\|_{W_p^{2,1}(\bar{Q}_T)} \leq C_P(T).$$

根据引理2可得

$$\begin{aligned} \|n\|_{W_p^{2,1}(\bar{Q}_T)} &\leq C_P(T), \|v\|_{W_p^{2,1}(\bar{Q}_T)} \leq \\ &C_P(T), \|u\|_{W_p^{2,1}(\bar{Q}_T)} \leq C_P(T). \end{aligned} \quad (72)$$

由 $W_p^{2,1}(\bar{Q}_T) \subset C^{1,\frac{1}{2}}(\bar{Q}_T)$, $\left(p > \frac{5}{2}\right)$, 有

$$\|c\|_{\infty} \leq C(T), \|n\|_{\infty} \leq C(T), \|u\|_{\infty} \leq C(T), \|v\|_{\infty} \leq C(T).$$

引理3得证.

由定理2, 引理3可证得本文的主要结论定理1.

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